# Brachistochrone of Entanglement for Spin Chains

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We analytically investigate the time-optimal unitary evolution of entanglement between indirectly coupled qubits in a trilinear Ising chain with unequal interaction couplings. The intermediate qubit is controlled via a local magnetic field. We find the time-optimal unitary evolution law, and we quantify residual entanglement via the two-tangle between the indirectly coupled qubits, for all possible sets of initial pure quantum states of a tripartite system. The integrals of the motion of the brachistochrone are determined by fixing the minimal time at which the residual entanglement is maximized. Entanglement plays a role for $W$ and $GHZ$ initial quantum states, and for the bi-separable initial state in which the indirectly coupled qubits have a nonzero value of the 2-tangle.

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## I. INTRODUCTION

The concept of entanglement is one of the key features distinguishing the quantum world from the classical world, as it captures those correlations which cannot have a classical origin [1]. It is one of the fundamental resources in quantum information and computation theory, teleportation, superdense coding, quantum cryptography, black hole physics etc Several measures of entanglement have been proposed for multipartite quantum systems in different contexts, for a review see, e.g., [2]-[5]. On the other hand, the speed of evolution for quantum systems is an important concept, not only for determining the theoretical limits at which quantum information can travel [6]-[7], but also for the practical task of building quantum computers capable of performing fast quantum algorithms before the ubiquitous and disruptive decoherence effects come into play. The importance of the connection between quantum entanglement, the speed of evolution of quantum systems and dynamical optimization problems has been discussed in [8]-[12]. Furthermore, quantum optimal control is also a fundamental subject, both theoretically and experimentally, in quantum computing and information (see, e.g., [13]). In particular, time-optimal quantum computation, where the cost to be optimized is the time to achieve a given quantum evolution is relevant for the design of fast elementary gates and provides a more physical ground to describe the complexity of quantum algorithms. A theoretical framework for the quantum brachistochrone (QB) was introduced in [14]. The QB [31] is based on a variational principle enforcing the time-optimal evolution of a quantum system whose Hamiltonian is subject to a set of constraints (e.g., a finite energy, certain qubit interactions are forbidden) and defines a boundary value problem with fixed initial and final quantum states (or unitary transformations). The QB has been studied for quantum state evolution in the case of pure [14] and mixed states [15], for the optimal realization of unitary transformations up to a given target quantum gate [16], and for the more realistic situation where the target can be reached within a finite, tolerable error (a fixed fidelity) [17]. Some authors [18]-[19] started the study of the role of quantum entanglement during the QB evolution of multipartite distinguishable systems in a pure quantum state, finding that the entanglement is pivotal to the QB evolution if at least two subsystems actively evolve. Efficient generation of random multipartite entangled states has been also analyzed with the aid of time-optimal unitary operations [20]. More recently, the authors of [21] found that genuine tripartite entanglement is necessary during the QB evolution of a set of three qubits in the pure state, except for the case in which less than three qubits attend evolution.

In this paper we discuss the role played by the entanglement in the QB formalism by considering the time-optimal unitary evolution of a tripartite system with a trilinear Ising interaction Hamiltonian with a fixed energy available and where the intermediate party can be controlled via a local, time dependent magnetic field. We point out that the time-optimal synthesis of interactions between qubits which are indirectly coupled via an intermediate qubit is a typical scenario in a wide array of promising (scalable) experimental realizations of quantum information processing (see, e.g., [22]-[26]). Moreover, the QB formalism naturally allows for the situation in which local coherent controls are assumed to time consuming, contrary to the requirements of zero time cost for local controls typical of the standard time-optimal quantum control methods [27], We concentrate our attention on the behavior of the residual entanglement between the indirectly coupled qubits at the end of the trilinear chain, as expressed by the so called 2-tangle [28]. We also con-

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sider all the possible initial quantum pure states for the tripartite system, i.e. completely separable, bi-separable, and with true tripartite entanglement (bipartite for $W$ and tripartite for $GHZ$ states). We then let the 2-tangle evolve along the general time-optimal quantum trajectory defined by the QB action principle, and we fix the integrals of the motion by imposing that entanglement reaches its maximum value in the shortest possible time. The minimal time for reaching such a maximum of the 2-tangle is a function of the ratio between the interaction couplings of the Ising Hamiltonian.

The paper is organized as follows. In Section II we review the main features of the QB formalism for the time-optimal synthesis of unitary quantum evolutions. In Section III we summarize the QB solution for the problem of a three-linear qubit system subject to an Ising interaction with unequal couplings and a local control on the intermediate qubit, when a finite energy is available. In Section IV we define the measure for the bipartite entanglement between the two indirectly coupled qubits of the chain and we introduce the main formulas for the 2-tangle and 3-tangle for all the possible sets of initial quantum states. Section V is devoted to the study of the entanglement evolution for these initial quantum states, and we define the optimal times and the analytical form of the quantum evolutions for which the 2-tangle between the indirectly coupled qubits most rapidly reach its maximum value. Finally, Section IV is devoted to the summary and discussion of our results.

## II. TIME-OPTIMAL UNITARY EVOLUTION

The goal is to determine the time-optimal way to generate the unitary evolution up to a certain $U_f$ (modulo physically irrelevant overall phases) by controlling an Hamiltonian $H(t)$ obeying the Schrödinger equation. We assume that $H$ is controllable and that only a finite energy is available in the experiment. This time-optimality problem may be formulated using the action [16]:

$$S(U,H;\alpha,\Lambda,\lambda_j) := \int_0^1 ds\,[N\alpha + L_S + L_C]\,, \qquad (1)$$

$$L_S := \langle \Lambda, i\tfrac{dU}{ds}U^\dagger - \alpha H\rangle, \qquad (2)$$

$$L_C := \alpha \sum_j \lambda_j f^j(H), \qquad (3)$$

where $\langle A,B\rangle := \mathrm{Tr}(A^\dagger B)$ and the Hermitian operator $\Lambda(s)$ and the real functions $\lambda_j(s)$ are Lagrange multipliers. The quantity $\alpha$ is the time cost, relating the parameter time $s$ and the physical time $t$ via $t := \int \alpha(s)ds$. Variation of $L_S$ by $\Lambda$ gives the Schrödinger equation:

$$i\frac{dU}{dt} = HU, \quad \text{or} \quad U(t) = \mathcal{T}e^{-i\int_0^t H dt}, \qquad (4)$$

where $\mathcal{T}$ is the time ordered product. Variation of $L_C$ by $\lambda_j$ leads to the constraints for $H$:

$$f_j(H) = 0. \qquad (5)$$

In particular, the finite energy condition for a system of $\log N$ qubits reads:

$$f_0(H) := \tfrac{1}{2}[\mathrm{Tr}(H^2) - N\omega^2] = 0, \qquad (6)$$

where $\omega$ is a constant.

From the variation of $S$ with respect to $H$ we get:

$$\Lambda = \lambda_0 H + \sum_{j\neq 0}\lambda_j \frac{\partial f_j(H)}{\partial H}, \qquad (7)$$

while from the variation of $S$ by $\alpha$ we obtain the normalization condition:

$$\mathrm{Tr}(H\Lambda) = N. \qquad (8)$$

Finally, variation of $S$ by $U$, use of eq. (7) and some elementary algebra give the *quantum brachistochrone equation*:

$$i\frac{d\Lambda}{dt} = [H,\Lambda], \qquad (9)$$

The *quantum brachistochrone* together with the constraints define a boundary-value problem for the evolution of the unitary operator $U(t)$ with fixed initial ($U(t=0)=1$, where 1 is the identity matrix) and final conditions ($U(t=T)=U_f$, where $T$ is the optimal time duration necessary to achieve the target gate $U_f$). It can be solved together with the constraint functions $f_j(H)$ to obtain $H_{\mathrm{opt}}(t)$. Then one integrates the Schrödinger equation (4) with $U(0)=1$ to get $U_{\mathrm{opt}}(t)$ and finally the integration constants in $H_{\mathrm{opt}}(t)$ can be fixed, e.g., by imposing that $U_{\mathrm{opt}}(T)$ equals a target $U_f$ modulo a global (physically irrelevant) phase.

## III. QB FOR A TRILINEAR ISING CHAIN

We now apply the general formalism of the QB to the case of a physical system of three qubits (labeled by a superscript $a\in\{1,2,3\}$) interacting via an Ising Hamiltonian and where the intermediate qubit is subject to a local and controllable magnetic field $B_i(t)$ ($i=x,y,z$):

$$H(t) := J_{12}\sigma_z^1\sigma_z^2 + J_{23}\sigma_z^2\sigma_z^3 + \vec{B}(t)\cdot\vec{\sigma}^2, \qquad (10)$$

where we have defined $\sigma_i^1\sigma_j^2 := \sigma_i\otimes\sigma_j\otimes 1$, $\sigma_i^2\sigma_j^3 := 1\otimes\sigma_i\otimes\sigma_j$, $\sigma_i^2 := 1\otimes\sigma_i\otimes 1$ and $\sigma_i$ are the Pauli operators. Introducing the ratio between the Ising interaction couplings $K := J_{23}/J_{12}$ and rescaling time as $\tau := J_{12}t$, the energy as $\hat{\omega} := \omega/J_{12}$ and the magnetic field as $\vec{B}_{\mathrm{opt}}(t) = J_{12}\hat{\vec{B}}_{\mathrm{opt}}(\tau)$, it can be shown [16]-[17] that the QB is solved by the following time-optimal magnetic field:

$$\hat{\vec{B}}_{\mathrm{opt}}(\tau) = \begin{pmatrix} \hat{B}_0\cos\theta(\tau) \\ \hat{B}_0\sin\theta(\tau) \\ \hat{B}_z \end{pmatrix}, \qquad (11)$$

where $\theta(\tau) := \hat{\Omega}\tau + \theta_0$ and $\hat{\Omega}$, $\theta_0$, $\hat{B}_0$ and $\hat{B}_z$ are integration constants. The magnetic field is precessing around the $z$-axis with the frequency $\hat{\Omega}$. Furthermore, the energy constraint explicitly reads $\hat{\vec{B}}^2 = \hat{\omega}^2 - (1+K^2) := \hat{\omega}_k^2$, so that we can reparameterize:

$$\hat{B}_0 := \hat{\omega}_k \cos\phi; \quad \hat{B}_z := \hat{\omega}_k \sin\phi, \tag{12}$$

where and $\phi \in [0, 2\pi]$.

One can then solve the Schrödinger equation (4) and find the time-optimal evolution operator [16]-[17]:

$$\begin{aligned} U_{\rm opt}(\tau) &= e^{-i\frac{\Omega\tau}{2}} A_D^{13}(\tau)|0\rangle_2\langle 0| \\ &\quad - ie^{i\frac{\Omega\tau}{2}-\theta(\tau)} B_0 S_D^{13}(\tau)|1\rangle_2\langle 0| + \text{H.c.}, \end{aligned} \tag{13}$$

where we have introduced the diagonal operators acting in the Hilbert space of qubits 1 and 3

$$A_D^{13}(\tau) := \text{Diag}[a_1(\tau), a_2(\tau), a_3(\tau), a_4(\tau)], \tag{14}$$
$$S_D^{13}(\tau) := \text{Diag}[s_1(\tau), s_2(\tau), s_3(\tau), s_4(\tau)]. \tag{15}$$

The latter operators depend upon the functions of time:

$$s_i(\tau) := \frac{\sin(\omega_i\tau)}{\omega_i}, \tag{16}$$
$$c_i(\tau) := \cos(\omega_i\tau), \tag{17}$$
$$a_i(\tau) := c_i(\tau) + i\, b_i s_i(\tau), \tag{18}$$

and on the constants:

$$\begin{aligned} \omega_i &:= \omega_k\sqrt{\cos^2\phi + b_i^2}, \\ b_i &:= \sin\phi + \frac{1}{\omega_k}\Big[(\delta_{i1}-\delta_{i4})(1+K) \\ &\quad + (\delta_{i2}-\delta_{i3})(1-K) - \frac{\hat{\Omega}}{2}\Big], \end{aligned} \tag{19}$$

where $i = 1, 2, 3, 4$ and $\delta_{ij}$ is the Kronecker symbol.

## IV. ENTANGLEMENT IN THE 1-3 SUBSYSTEM

For a tripartite quantum system in a pure state $|\psi\rangle = \sum_{i=0}^{7} a_i|i\rangle$ expanded in the basis $\{|0\rangle := |000\rangle, |1\rangle := |001\rangle, |2\rangle := |010\rangle, |3\rangle := |011\rangle, |4\rangle := |100\rangle, |5\rangle := |101\rangle, |6\rangle := |110\rangle, |7\rangle := |111\rangle\}$ the entanglement between two of its subsystems, e.g. 1 and 3, may be quantified by the 2-tangle [5], [28]:

$$\tau_{13} := 2[\text{Det}[\rho_1] - \text{Det}[\rho_2] + \text{Det}[\rho_3] - |\text{HypDet}(a)|], \tag{20}$$

where $\rho_i$ $(i = 1, 2, 3)$ are the reduced density matrices

$$\begin{aligned} \rho_1 &:= \text{Tr}_{23}(|\psi\rangle\langle\psi)|), \\ \rho_2 &:= \text{Tr}_{13}(|\psi\rangle\langle\psi)|), \\ \rho_3 &:= \text{Tr}_{12}(|\psi\rangle\langle\psi)|), \end{aligned} \tag{21}$$

and HypDet$(a)$ is Cailey's hyperdeterminant for the matrix of the coefficients $a_i$s (see eq. (46) in the Appendix).

Furthermore, one may also define the 3-tangle:

$$\tau_{123} := |\text{HypDet}(a)|, \tag{22}$$

which describes the amount of tripartite entanglement between all the spins.

We now consider the time optimal evolution of an arbitrary initial pure state $|\psi(0)\rangle$ driven by the evolution operator (13), and we study the behavior of the bipartite entanglement between the indirectly coupled qubits 1 and 3. We are interested, in particular, about determining the optimal time $\tau_*$ and the integration constants $B_0, B_z, \Omega, \theta_0$ for which the 2-tangle $\tau_{13}$ is maximized.

As it is well known, pure states of a tripartite quantum system can be classified into equivalence classes under local operations and classical communication (LOCC), distinguishing the degree of entanglement between its subsystems [29]. In particular, we will consider the representatives of each of these classes as a possible initial quantum state:

a) initial completely separable states, for which all 2-tangles and the 3-tangle vanish:

$$|\psi_(0)\rangle = |\psi_S\rangle := |000\rangle; \tag{23}$$

b) initial bi-separable states, with bipartite entanglement, for which one of the 2-tangles is nonzero while the other 2-tangles and the 3-tangle are zero:

$$|\psi(0)\rangle = |\psi_{BS1}\rangle := \frac{1}{\sqrt{2}}(|001\rangle + |010\rangle), \tag{24}$$
$$|\psi(0)\rangle = |\psi_{BS1}\rangle := \frac{1}{\sqrt{2}}(|001\rangle + |100\rangle), \tag{25}$$
$$|\psi(0)\rangle = |\psi_{BS1}\rangle := \frac{1}{\sqrt{2}}(|010\rangle + |100\rangle); \tag{26}$$

c) initial Werner (W) states, with full bipartite entanglement, for which all 2-tangles vanish while the 3-tangle is nonzero:

$$|\psi(0)\rangle = |\psi_W\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle); \tag{27}$$

d) initial Greenberger-Horne-Zeilinger (GHZ) states with maximal tripartite entanglement, for which all 2-tangles and the 3-tangle are nonzero:

$$|\psi(0)\rangle = |\psi_{GHZ}\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{28}$$

The time-optimal evolution operator $U_{\rm opt}(\tau)$, eq. (13), drives the initial states according to

$$|\psi(\tau)\rangle = U_{\rm opt}(\tau)|\psi(0)\rangle = \sum_{i=0}^{7} a_i(\tau)|i\rangle. \tag{29}$$

For each of the possible classes of initial states, we compute the nonzero time-dependent amplitudes, which are

explicitly given by formulas (51)-(56) of the Appendix. Then it is easy to check, substituting these formulas into (46) and (50), and then into (20) and (22), that both the $\tau_{13}(\tau)$ and the $\tau_{123}(\tau)$ tangles are always zero during the whole time-optimal evolution of a 3-qubit state which is initially fully separable or a bi-separable state of type $B1$ and $B3$ (i.e., when initially there is no entanglement between the indirectly coupled qubits 1 and 3).

On the other hand, when there is some initial entanglement between qubits 1 and 3, then we have a non trivial time-optimal evolution of the $\tau_{13}(\tau)$ and $\tau_{123}(\tau)$ tangles. For example, when the initial state belongs to the class $B2$ of bi-separable states, from eqs. (20), (22), (46), (50) and (53) and we get the following time-optimal evolutions:

$$\tau_{13_{B2}}(\tau) = |a_2^* a_3 + \hat{B}_0^2 s_2 s_3|^2, \quad (30)$$

$$\tau_{123_{B2}}(\tau) = \frac{\hat{B}_0^2}{4}|a_2 s_3 - a_3 s_2|^2 = 1 - \tau_{13_{B2}}(\tau). \quad (31)$$

The case of an initial $W$ state is similar and we find that $\tau_{13_W}(\tau) = (4/9)\tau_{13_{B2}}(\tau)$ and $\tau_{123_W}(\tau) = (4/9)\tau_{123_{B2}}(\tau)$. Finally, for the class of fragile, fully entangled $GHZ$ states the time-optimal evolution of the tangles is found to be given by:

$$\tau_{13_{GHZ}}(\tau) = \hat{B}_0^2 |a_1 s_4 - a_4 s_1|^2, \quad (32)$$

$$\tau_{123_{GHZ}}(\tau) = \frac{1}{4}|a_1^* a_4 + \hat{B}_0^2 s_1 s_4|^2 = 1 - \tau_{13_{GHZ}}(\tau). \quad (33)$$

## V. TIME-OPTIMAL EVOLUTION OF ENTANGLEMENT

The behavior of the time-optimal evolution of the tangles $\tau_{13}, \tau_{123}$, which witness the entanglement between qubits 1 and 3, can be studied in more details by explicit use of the formulas (16), (18) and (19) into equations (30)-(33). As the tangles are always related by $\tau_{13}(\tau) = 1 - \tau_{123}(\tau)$, we limit ourselves to the study of the $\tau_{13}$ tangle.

For the case of an initial bi-separable state of class $B2$ (and similarly for initial states of the class $W$), from (18) and (30) we obtain:

$$\begin{aligned}\tau_{13_{B2}}(\tau) = {} & [c_2 c_3 + \hat{\omega}_k^2(\cos^2\phi + b_2 b_3) s_2 s_3]^2 \\ & + \hat{\omega}_k^2 (b_2 s_2 c_3 - b_3 s_3 c_2)^2. \end{aligned} \quad (34)$$

We now proceed by further optimizing the tangle $\tau_{13_{B2}}(\tau)$ as a function of the time $\tau$ and of the unknown constants of the motion for the time-optimal quantum trajectory, i.e. $\hat{B}_0, \hat{B}_z, \hat{\Omega}$, for a given ratio of the couplings $K$ and for fixed energy $\hat{\omega}$. In particular, we look for the time $\tau_*$ at which the tangle first reaches its maximum value $\tau_{13}|_{\max} = 1$ by imposing that the partial derivatives of the tangle with respect to $\tau$ and the other constants of the motion (expressed in terms of $\phi$ and $\hat{\Omega}$) are zero and that the determinant of the Hessian matrix at this point is negative. After a long and tedious but simple algebra we find that "optimal" time at which $\tau_{13_{B2}}(\tau)$ first reaches its maximum depends on the value of the available energy $\hat{\omega}^2$ and on the ratio $K$ between the couplings in the Ising Hamiltonian. In more details, we have that:

$$\tau_{*_{B2}} = \frac{\sqrt{3}}{4}\frac{\pi}{|1-K|}, \quad (35)$$

when $1 < \hat{\omega}^2 < 29/16$ and $|K| < K_{1+}$, or when $\hat{\omega}^2 > 29/16$ and $K_{1-} < K < K_{2-}$ or $K_{2+} < K < K_{1+}$, where we have defined $K_{1\pm} := \pm\sqrt{\hat{\omega}^2 - 1}$ and $K_{2\pm} := (1/4)[13/4 \pm \sqrt{3(\hat{\omega}^2 - 29/16)}]$. The duration (43) of the optimal quantum evolution for the bi-partite entanglement is minimal for a ratio of the Ising couplings $K \to K_{1\pm}$, and maximal for a ratio of the Ising couplings $K \to K_{2\pm}$. Furthermore, in this case we obtain the optimal magnitudes and frequency for the magnetic field as:

$$|\hat{B}_{0_{B2}}| = \frac{2}{\sqrt{3}}|K-1|, \quad (36)$$

$$|\hat{B}_{z_{B2}}| = \sqrt{\hat{\omega}^2 - \frac{7}{3}K^2 + \frac{8}{3}K - \frac{49}{21}}, \quad (37)$$

$$\hat{\Omega}_{B2} = 2\left[K - 1 \pm \sqrt{\hat{\omega}^2 - \frac{7}{3}K^2 + \frac{8}{3}K - \frac{49}{21}}\right]. \quad (38)$$

Instead, we have that:

$$\tau_{*_{B2}} = \frac{\pi}{\sqrt{\hat{\omega}^2 - 2K}} \quad (39)$$

when $\hat{\omega}^2 > 29/16$ and $K_{2-} < K < K_{2+}$. Within this range of allowed energies and $K$, the duration of the optimal quantum evolution for the bi-partite entanglement is minimal for a ratio of the Ising couplings $K \to K_{2-}$, and maximal for a ratio of the Ising couplings $K \to K_{2+}$. In this case, the optimal values for the magnetic field are:

$$|\hat{B}_{0_{B2}}| = \sqrt{\hat{\omega}^2 - (1 + K^2)}, \quad (40)$$

$$|\hat{B}_{z_{B2}}| = \hat{\Omega} = 0. \quad (41)$$

In Figs. 1-2 we plot the 2-tangle and the 3-tangle as a function of time when the quantum system is in the initial bi-separable state $B2$ and for the value of energy $\hat{\omega} = \sqrt{6}$. For this value of energy, we can compute $K_{1\pm} = \pm 2.24$ and $K_{2+} \simeq 1.70$, $K_{2-} \simeq -0.007$. To exemplify the values of the couplings, we two models proposed in [25] for nuclear magnetic resonance (NMR) experiments, i.e.: a) the $H-N-H$ chain in the molecule of ethanamide, for which $J_{12} = J_{23} \simeq 88.05$ Hz and therefore $K = 1$; b) the $P-F-H$ chain in the molecule of diethylfluoromethylphosphonate, for which $J_{12} \simeq 46$ Hz, $J_{23} \simeq 73.1$ Hz and therefore $K = 1.59$. For both $K = 1$ and $K = 1.59$ we are in the situation depicted by formula (39), and the the law of quantum evolution of the tangle is explicitly given by $\tau_{13_{B2}}(\tau) = 1 - 4[(1-K)^2\omega_k^2/(\hat{\omega}^2 - 2)^2]\sin^4[\sqrt{\hat{\omega}^2 - 2K}\tau]$.

We notice that the case of equal Ising couplings, i.e. case a) with $K = 1$, residual entanglement between the indirectly coupled qubits is constant and always maximal, equal to one, while the tripartite entanglement is always zero. For the case of unequal couplings, instead, the bipartite entanglement has a periodic behavior, starting from the maximum equal to one at $\tau = 0$, reaching a minimum of approximately 0.57 at $\tau = \tau_{*_{B2}}/2$, rising again to the maximum of one at $\tau = \tau_{*_{B2}}$ and so on.

Let us now turn to the case of an initial state belonging to the $GHZ$ class. From (18) and (32) and we obtain:

$$\tau_{13_{GHZ}}(\tau) = \hat{\omega}_k^2 \cos^2\phi[(c_1 s_4 - c_4 s_1)^2 + \hat{\omega}_k^2 (b_4 - b_1)^2 (s_1 s_4)^2]. \quad (42)$$

In this case, the "optimal" time at which $\tau_{13_{GHZ}}(\tau)$ first reaches its maximum is found to be:

$$\tau_{*_{GHZ}} = \frac{\sqrt{2}}{4}\frac{\pi}{|1+K|}, \quad (43)$$

provided that $\hat{\omega}^2 > 3/2$ and $K_- < K < K_+$, where we have defined $K_\pm := (1/2)[-1 \pm \sqrt{2\hat{\omega}^2 - 3}]$. For the $GHZ$ initial states, the duration of the optimal quantum evolution for the bi-partite entanglement is minimal for a ratio of the Ising couplings $K \to K_+$, and maximal for a ratio of the Ising couplings $K \to K_-$ (and it diverges if $\hat{\omega}^2 = 2$, and $K \to K_- = -1$). The associated optimal magnitudes and frequency for the magnetic field in the $GHZ$ case are:

$$|\hat{B}_{0_{GHZ}}| = |1+K|, \quad (44)$$

$$|\hat{B}_{z_{GHZ}}| = \frac{|\hat{\Omega}_{GHZ}|}{2} = \sqrt{\hat{\omega}^2 - 2(K^2 + K + 1)}. \quad (45)$$

In Figs. 3-4 we plot the 2-tangle and the 3-tangle as a function of time when the quantum system is in the initial $GHZ$ state and for the value of energy $\hat{\omega} = \sqrt{14}$. For this energy we have that $K_{2-} = -3$ and $K_{2+} = 2$. Therefore, we can still use the NMR models a) and b) of Figs. 1-2, with $K = 1$ and $K = 1.59$, respectively. Here the explicit analytical formula for the time-optimal evolution of the entanglement is given by $\tau_{13_{B2}}(\tau) = \sin^4[\sqrt{2}(1+K)\tau]$. Now the periodical behavior is present for both values of $K$, and the system oscillates between the initial maximal bipartite entanglement and zero entanglement.

## VI. DISCUSSION

We have analytically investigated the the problem of the time-optimal unitary evolution of entanglement between indirectly coupled qubits in a trilinear Ising chain with unequal interaction couplings, where the intermediate qubit is coherently controlled via a local magnetic field. The entanglement is quantified by the 2-tangle between the two qubits at the end of the chain. Using the formalism of the QB with the constraint of a fixed energy available, we have found the time-optimal unitary evolution law for the Ising Hamiltonian plus the local control and we substituted it in the formula for the 2-tangle. The initial boundary condition for the QB is chosen among all possible sets of tripartite quantum pure states, i.e. fully separable, bi-separable, Werner and $GHZ$ states. The integrals of the motion in the QB are determined imposing that the 2-tangle reaches its maximum in the shortest time possible, which we call $\tau_*$. Entanglement is found to have a non trivial role during the time-optimal unitary evolutions of $W$ and $GHZ$ initial quantum states, and of the bi-separable initial state in which the indirectly coupled qubits have a nonzero value of the 2-tangle. The optimal time $\tau_*$ also sets the time-scale for the duration of the significant role of the entanglement, and it is a function of the ratio $K$ between the interaction couplings in the Ising Hamiltonian and of the energy available in the experiment. The monogamy of entanglement shows neatly in the anti-correlation of the tripartite entanglement, quantified by the 3-tangle, with the bipartite entanglement, quantified by the 2-tangle. It is interesting to extend the analysis presented here to the case when coherent control is possible on all the qubits in the chain, to longer chains of qubits, and to study the QB evolution of the truly non classical correlations via their measure, quantum discord [30].

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## VII. APPENDIX

For a tripartite system 123 in a pure state $|\psi\rangle = \sum_{i=0}^{7} a_i |i\rangle$ expanded in the basis $\{|0\rangle := |000\rangle, |1\rangle := |001\rangle, |2\rangle := |010\rangle, |3\rangle := |011\rangle, |4\rangle := |100\rangle, |5\rangle := |101\rangle, |6\rangle := |110\rangle, |7\rangle := |111\rangle\}$, Cailey's hyperdeterminant for the matrix of the coefficients $a_i$s is defined as

$$\begin{aligned} \mathrm{HypDet}(a) := {} & [(a_0a_7)^2 + (a_1a_6)^2 + (a_2a_5)^2 + (a_3a_4)^2] \\ & - 2[(a_0a_7 + a_1a_6)(a_2a_5 + a_3a_4) \\ & + a_0a_1a_6a_7 + a_2a_3a_4a_5] \\ & + 4(a_0a_3a_5a_6 + a_1a_2a_4a_7). \end{aligned} \quad (46)$$

More explicitly, since

$$\begin{aligned}\mathrm{Det}(\rho_1) = {}& |a_0|^2(|a_5|^2+|a_6|^2+|a_7|^2)\\ &+|a_1|^2(|a_4|^2+|a_6|^2+|a_7|^2)\\ &+|a_2|^2(|a_4|^2+|a_5|^2+|a_7|^2)\\ &+|a_3|^2(|a_4|^2+|a_5|^2+|a_6|^2)\\ &-2\mathrm{Re}(a_0a_5a_1^*a_4^*+a_0a_6a_2^*a_4^*\\ &+a_0a_7a_3^*a_4^*+a_1a_6a_2^*a_5^*\\ &+a_1a_7a_3^*a_5^*+a_2a_7a_3^*a_6^*), \end{aligned} \tag{47}$$

$$\begin{aligned}\mathrm{Det}(\rho_2) = {}& |a_0|^2(|a_3|^2+|a_6|^2+|a_7|^2)\\ &+|a_1|^2(|a_2|^2+|a_6|^2+|a_7|^2)\\ &+|a_4|^2(|a_4|^2+|a_3|^2+|a_7|^2)\\ &+|a_5|^2(|a_4|^2+|a_3|^2+|a_6|^2)\\ &-2\mathrm{Re}(a_0a_3a_1^*a_2^*+a_0a_6a_2^*a_4^*\\ &+a_0a_7a_2^*a_5^*+a_1a_6a_3^*a_4^*\\ &+a_1a_7a_3^*a_5^*+a_4a_7a_5^*a_6^*), \end{aligned} \tag{48}$$

and

$$\begin{aligned}\mathrm{Det}(\rho_3) = {}& |a_0|^2(|a_3|^2+|a_5|^2+|a_7|^2)\\ &+|a_2|^2(|a_1|^2+|a_5|^2+|a_7|^2)\\ &+|a_4|^2(|a_1|^2+|a_3|^2+|a_7|^2)\\ &+|a_6|^2(|a_1|^2+|a_3|^2+|a_5|^2)\\ &-2\mathrm{Re}(a_0a_3a_1^*a_2^*+a_0a_5a_1^*a_4^*\\ &+a_0a_7a_1^*a_6^*+a_2a_5a_3^*a_4^*\\ &+a_2a_7a_3^*a_6^*+a_4a_7a_5^*a_6^*), \end{aligned} \tag{49}$$

we obtain:

$$\begin{aligned}\mathrm{Det}(\rho_1)-\mathrm{Det}(\rho_2)+\mathrm{Det}(\rho_3) = {}& 2(|a_0|^2|a_5|^2+|a_1|^2|a_4|^2\\ &+|a_2|^2|a_7|^2+|a_3|^2|a_6|^2)\\ &-2\mathrm{Re}[a_0a_7a_1^*a_6^*+a_2a_5a_3^*a_4^*\\ &-(a_0a_7-a_1a_6)(a_2^*a_5^*-a_3^*a_4^*)]\\ &-4\mathrm{Re}(a_0a_5a_1^*a_4^*+a_2a_7a_3^*a_6^*). \end{aligned} \tag{50}$$

Given the time-optimal unitary evolution (29), we can compute the following nonzero time-dependent amplitudes (all modulo $\exp[-i\Omega\tau/2]$):

a) representative of the class of fully separable initial states, $|\psi_S\rangle$:

$$\begin{aligned}(a_S)_0(\tau) &= a_1^*,\\ (a_S)_2(\tau) &= -i\hat{B}_0 e^{i\theta} s_1; \end{aligned} \tag{51}$$

b1) representative of the class of bi-separable initial states, $|\psi_{BS1}\rangle$:

$$\begin{aligned}(a_{B1})_0(\tau) &= -i\frac{\hat{B}_0}{\sqrt{2}}e^{-i\theta_0}s_1,\\ (a_{B1})_1(\tau) &= \frac{a_2^*}{\sqrt{2}},\\ (a_{B1})_2(\tau) &= \frac{e^{i\hat{\Omega}\tau}}{\sqrt{2}}a_1,\\ (a_{B1})_3(\tau) &= -i\frac{\hat{B}_0}{\sqrt{2}}e^{i\theta}s_2; \end{aligned} \tag{52}$$

b2) representative of the class of bi-separable initial states, $|\psi_{B2}\rangle$:

$$\begin{aligned}(a_{B2})_1(\tau) &= \frac{a_2^*}{\sqrt{2}},\\ (a_{B2})_3(\tau) &= -i\frac{\hat{B}_0}{\sqrt{2}}e^{i\theta}s_2,\\ (a_{B2})_4(\tau) &= \frac{a_3^*}{\sqrt{2}},\\ (a_{B2})_6(\tau) &= -i\frac{\hat{B}_0}{\sqrt{2}}e^{i\theta}s_3; \end{aligned} \tag{53}$$

b3) representative of the class of bi-separable initial states, $|\psi_{B3}\rangle$:

$$\begin{aligned}(a_{B3})_0(\tau) &= -i\frac{\hat{B}_0}{\sqrt{2}}e^{-i\theta_0}s_1,\\ (a_{B3})_2(\tau) &= \frac{e^{i\hat{\Omega}\tau}}{\sqrt{2}}a_1,\\ (a_{B3})_4(\tau) &= \frac{a_3^*}{\sqrt{2}},\\ (a_{B3})_6(\tau) &= -i\frac{\hat{B}_0}{\sqrt{2}}e^{i\theta}s_3; \end{aligned} \tag{54}$$

c) representative of the class of Werner initial states, $|\psi_W\rangle$:

$$\begin{aligned}(a_W)_0(\tau) &= -i\frac{\hat{B}_0}{\sqrt{3}}e^{-i\theta_0}s_1,\\ (a_W)_1(\tau) &= \frac{a_2^*}{\sqrt{3}},\\ (a_W)_2(\tau) &= \frac{e^{i\hat{\Omega}\tau}}{\sqrt{3}}a_1,\\ (a_W)_3(\tau) &= -i\frac{\hat{B}_0}{\sqrt{3}}e^{i\theta}s_2,\\ (a_W)_4(\tau) &= \frac{a_3^*}{\sqrt{3}},\\ (a_W)_6(\tau) &= -i\frac{\hat{B}_0}{\sqrt{3}}e^{i\theta}s_3; \end{aligned} \tag{55}$$

d) representative of the class of GHZ initial states, $|\psi_{GHZ}\rangle$:

$$\begin{aligned}(a_{GHZ})_0(\tau) &= \frac{a_1^*}{\sqrt{2}},\\ (a_{GHZ})_2(\tau) &= -i\frac{\hat{B}_0}{\sqrt{2}}e^{i\theta}s_1,\\ (a_{GHZ})_5(\tau) &= -i\frac{\hat{B}_0}{\sqrt{2}}e^{-i\theta_0}s_4,\\ (a_{GHZ})_7(\tau) &= \frac{e^{i\hat{\Omega}\tau}}{\sqrt{2}}a_4.\end{aligned} \tag{56}$$


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[31] From the Greek "$\beta\rho\alpha\chi\iota\sigma\tau o\zeta$", i.e., fast, and "$\chi\rho o\nu o\zeta$", i.e time.


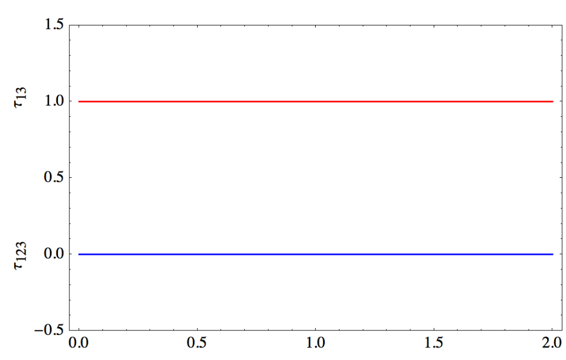


FIG. 1: The 2-tangle (red curve) and the 3-tangle (blue curve) as a function of time for the $B2$ initial states and the coupling ratio $K = 1$.

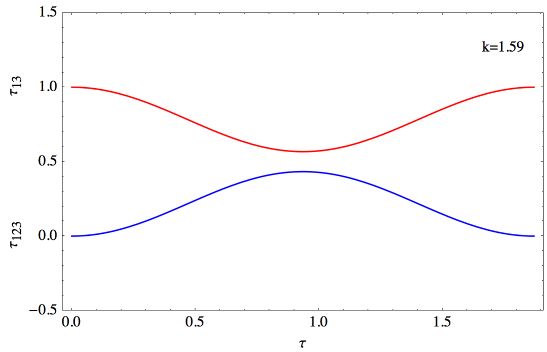


FIG. 2: The 2-tangle (red curve) and the 3-tangle (blue curve) as a function of time for the $B2$ initial states and the coupling ratio $K = 1.59$.

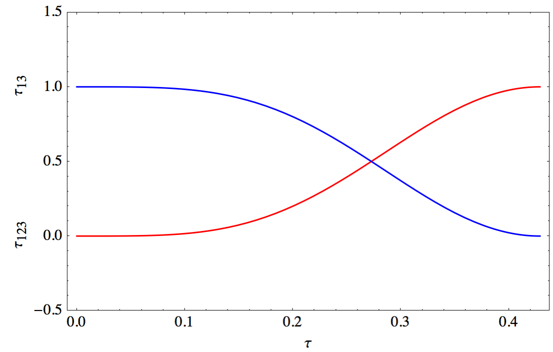


FIG. 4: The 2-tangle (red curve) and the 3-tangle (blue curve) as a function of time for the $GHZ$ initial states and the coupling ratio $K = 1.59$.

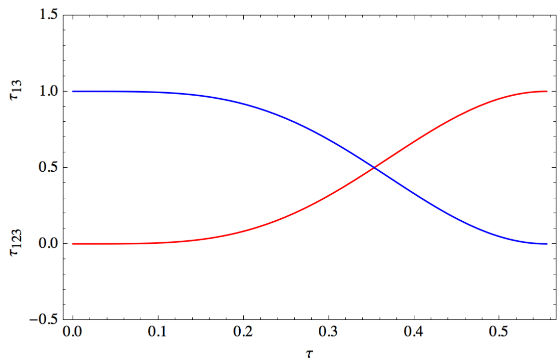


FIG. 3: The 2-tangle (red curve) and the 3-tangle (blue curve) as a function of time for the $GHZ$ initial states and the coupling ratio $K = 1$.